Moving Fourier analysis for locally stationary processes with the bootstrap in view*

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Abstract

We introduce a moving Fourier transformation for locally stationary time series, which captures the time-varying spectral density in a similar manner as the classical Fourier transform does for stationary time series. In particular, the resulting Fourier coefficients as well as moving local periodograms are shown to be (almost all) asymptotically uncorrelated. The moving local periodogram is obtained by thinning the local periodogram to avoid multiple information present at different but close points in time. We obtain consistent estimators for the local spectral density at each point in time by smoothing the moving local periodogram. Furthermore, the moving Fourier coefficients respectively periodograms are well suited to adapt stationary frequency-domain bootstrap methods to the locally stationary case. For the wild TFT-bootstrap it is shown that the corresponding bootstrap covariance of a global locally stationary bootstrap samples captures the time varying covariance structure of the underlying locally stationary time series correctly. Furthermore, this bootstrap in addition to adaptations of other frequency domain bootstrap methods are used in a simulation study to obtain uniform confidence bands for the time-varying autocorrelation at lag one. Finally, this methodology is applied to a wind data set.

Keywords: locally stationary time series, Fourier analysis, frequency domain bootstrap, local spectral density estimation

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1 Introduction

Many longer time series cannot be described by stationary models sufficiently well, while those models often describe the statistical behavior locally quite well. One

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important concept allowing for such effects was formally introduced and analyzed by Dahlhaus (1997), which allows in particular for valid asymptotic theory and is developed with spectral methods in mind. In this work we develop a modification of the Fourier transform in this locally stationary framework – the so called moving local Fourier transform - which enables us to transfer the local structure of the data from the time domain to the frequency domain, yet preserving the convenient property of the resulting Fourier coefficients being uncorrelated in the frequency domain. The local periodogram that has already previously been discussed in the literature is very convenient if one is only interested in the local structure of the time series, however, not very suited for global theory. More precisely, the statistical properties of the local periodogram around a given time point are equivalent to the periodogram of stationary time series: In particular periodogram ordinates are asymptotically independent and exponentially distributed with a parameter depending on the local spectrum at that point in time. On the other hand, if one combines all the local periodograms for all time points statistical information is duplicated as the statistical properties of the local periodogram centered around one time point and the time point next to it will be almost identical. In particular, this results in very complicated dependency relations between the local periodogram ordinates around one point and the local periodogram ordinates around a neighboring point. The moving local Fourier transform, that we propose in this paper, essentially thins out this information making use of the fact that statistical properties will not change much from one time point to another. To this end, we only calculate the local Fourier coefficients (periodogram) at one frequency, then calculate the next frequency based on a window shifted by one. We will prove that this transformation essentially results in asymptotically independent and normally distributed (exponentially in case of the corresponding periodograms) moving Fourier coefficients. We then prove asymptotic consistency of corresponding estimators for the spectrum.

For stationary time series, frequency domain bootstrap methods have proven quite useful in many situations Paparoditis (2002). Using the local Fourier transform (related to the local periodogram) for a local bootstrap works fine as long as it is only used for inference of the local structure such as the local autocorrelation (see Sergides and Paparoditis (2009)). However, in order to obtain a global locally stationary bootstrap sample, the full dependency between local Fourier coefficients corresponding to different time points needed to be mimicked correctly by the bootstrap which seems a very difficult task. For this reason Kreiss and Paparoditis (2015) use a global Fourier transform to adapt the hybrid wild bootstrap to the locally stationary situation where the local periodogram is merely needed to estimate the local spectra. In this paper, we propose a different approach by using our moving local Fourier transform. This is quite close in spirit to the original idea of frequency domain bootstrapping for stationary time series because the moving local Fourier coefficients have similar statistical properties globally (i.e. they are asymptotically independent and normal) as stationary (global) Fourier coefficients. Using a similar moving local Fourier back-transformation we are able to transform the bootstrap Fourier coefficients back into the time domain. Consequently, we are able to construct a bootstrap imitation of the original locally stationary time series. We will focus on an adaptation of the TFT bootstrap by Kirch and Politis (2011) to locally stationary time series to obtain theoretic results but will probe in simulations also extensions to other recent frequency domain bootstrap methods for stationary time series such as the wild hybrid bootstrap (Kreiss and Paparoditis (2012b)) as well as the autoregressive aided periodogram bootstrap (Kreiss and Paparoditis (2003)). One main advantage of getting a locally stationary bootstrap sample in the time domain is the possibility to construct **uniform** confidence bounds (say for the local autocorrelation of order one). While a theoretic analysis of such a procedure is postponed to future work, we show that this works nicely in simulations.

The paper is organized as follows: In Section 2 we introduce the moving local Fourier transform (MLFT) as well as the corresponding backtransform. In the formal setting of locally stationary time series we obtain asymptotic statistical properties of the MLFT in Section 2.4 that are similar to corresponding properties of the stationary Fourier transform such as asymptotic uncorrelatedness of (almost all) moving Fourier coefficients. The proofs are based on an approximation of the MLFT by the MLFT of independent random variables which is formally introduced in Section 2.5. In Section 2.6 the moving local Fourier transform is used to estimate the time varying spectral density of the underlying process. Section 3 is devoted to introducing how several stationary frequency domain bootstrap methods can be adopted to locally stationary time series based on the MLFT, where for the TFT bootstrap it is shown that the time varying covariance structure is correctly captured by a global bootstrap sample. In Section 4 these bootstrap methods are applied in a simulation study to obtain uniform confidence bounds for the time varying autocorrelation at lag 1 as well as to a wind data set formerly investigated by several authors. Finally, the proofs are given in Section 5.

2 Moving local Fourier transform (MLFT)

The main idea behind locally stationary processes is that the statistical structure of many data sets only changes slowly with time so that in an environment of each point the time series can be considered to be approximately stationary. To obtain theoretic results we work with the formal definition of Dahlhaus (1997, 2003) (see Section **??** for details). However, this vague statement is sufficient to understand the idea behind the moving local Fourier transform that is introduced in this section.

Recently, Sergides and Paparoditis (2009) proposed to use the local periodogram for bootstrapping certain quantities such as ratio statistics at a given time point $1 \leqslant t_0 \leqslant T$. To this end, they calculate the periodogram based on the time series of an appropriate environment of this point only. This is reasonable due to the assumption that within such an environment the time series say $X_{t_0-m}, \ldots, X_{t_0+m}$ behaves approximately stationary. Statistical properties of such periodograms are then essentially the same as for the usual periodogram for stationary time series such that they can be used to get a consistent estimator of the local spectral density. This works very well if the environment is chosen appropriately and if the statistical quantity of interest is only a local quantity. As soon as one is interested in global quantities as is e.g. necessary for a global locally stationary bootstrap the problem arises that the local periodogram of a shifted stretch $X_{t_0-m+1}, \ldots, X_{t_0+m+1}$ will carry essentially the same statistical information but exhibit strong dependence with the local periodogram of the orginal stretch $X_{t_0-m}, \ldots, X_{t_0+m}$. This dependence makes statistical analysis much more difficult and appears to pose unsurmountable difficulties in cases where one tries to bootstrap this periodogram ordinates hence has to mimic the dependency structure between the periodogram ordinates of different (!) stretches correctly.

On the other hand, the information is more or less duplicated due to the slowly changing nature of a locally stationary time series if stretches are close together. It is this latter observation that we will make use of in the following construction of moving local Fourier coefficients.

2.1 Definition

For ease of notation we assume we have observed $X_{-m+1}, \ldots, X_1, \ldots, X_T, \ldots, X_{T+m}$ (where $X_j = X_{j,T}$ to allow for a formal definition of a slowly moving time series structure).

Define the discrete Fourier coefficients of the stretch $X_{t_0-m},\ldots,X_{t_0+m}$ of length $2m+1$ around time point t_0 at Fourier frequencies $\lambda_k := \frac{2\pi k}{2m+1}$, $k=1,\ldots,m$, (depending on the length $2m+1$) by

$$\mathcal{F}(X_{t_0-m},\ldots,X_{t_0+m};\lambda_k) \quad = \quad \frac{1}{\sqrt{2m+1}}\sum_{l=0}^{2m} X_{l+t_0-m}e^{-il\lambda_k}.$$

As already pointed out by applying the same transformation to the shifted stretch $X_{t_0-m+1},\ldots,X_{t_0+m+1}$ we obtain essentially the same statistical information as the local structure of the shifted stretch will not have changed much.

In order to avoid this duplication, we calculate only one Fourier coefficient belonging to one frequency for each stretch but use the subsequent Fourier frequency for the adjacent stretch. More precisely, we calculate only the Fourier coefficient at frequency $\lambda_{\mathrm{mod}(t_0)}$ for a stretch centered around time point t_0, where

$$\mathrm{mod}(j) := \begin{cases} m, & \text{if } m \text{ is a factor of } j\in\mathbb{Z}, \\ j \mod (m), & \text{else} \end{cases} \tag{2.1}$$

varies between 1 and m.

Definition 2.1 (Moving Fourier coefficients). *The moving Fourier coefficients MF_k (of order m), $1\leqslant k\leqslant T$, of $X_{-m+1},\ldots,X_{T+m}$ are defined by*

$$MF_k := \mathcal{F}(X_{k-m,T},\ldots,X_{k+m,T};\lambda_{\mathrm{mod}(k)}) = \frac{1}{\sqrt{2m+1}}\sum_{\nu=0}^{2m} X_{\nu+k-m,T}e^{-i\nu\lambda_{\mathrm{mod}(k)}}, \tag{2.2}$$

where $\lambda_{mod(k)} := \frac{2\pi\, mod(k)}{2m+1}$.

Similarly to the discrete Fourier transform for stationary sequences this results in an approximately uncorrelated sequence of length T with variance proportional to the local spectral density at frequency mod(k) (confer Theorems 2.1 and 2.2 below). As such it can be of great use for a statistical analysis of the time series with a potential for obtaining uniform results. In particular, it can be used for bootstrap purposes which is the main application in this work.

The main problem remains how one can extract a complete set of Fourier coefficients for the local structure at some given time point t_0. Here, the idea is to collect the m closest Fourier coefficients, i.e. $MF_{t_0-\lfloor m/2\rfloor},\ldots,MF_{t_0+\lceil m/2\rceil-1}$, containing the local information at all frequencies λ_k, $k=1,\ldots,m$, although not necessarily in the correct order. Bringing them in the correct order of frequency results in the following definition of moving local Fourier coefficients.

Definition 2.2 (Moving local Fourier coefficients). *The moving local Fourier coefficients (of order m) at time t_0 for frequencies λ_l, $l=1,\ldots,m$, are given by*

$$MF_{t_0}(\lambda_l) := \begin{cases} MF_{t_0-\mathrm{mod}(t_0)+l-m}, & l-\mathrm{mod}(t_0)\geqslant\lceil m/2\rceil, \\ MF_{t_0-\mathrm{mod}(t_0)+l}, & -\lfloor m/2\rfloor\leqslant l-\mathrm{mod}(t_0)<\lceil m/2\rceil, \\ MF_{t_0-\mathrm{mod}(t_0)+l+m}, & l-\mathrm{mod}(t_0)<-\lfloor m/2\rfloor. \end{cases}$$

Furthermore,

$$MF_{t_0}(\lambda_{2m+1-j}) = MF_{t_0}(-j) := \overline{MF_{t_0}(\lambda_j)},\ j=0,\ldots,m, \quad MF_{t_0}(\lambda_0) := 0.$$

The moving local periodogram MI_{t_0} at time t_0 is then defined by

$$MI_{t_0}(\lambda_l) := |MF_{t_0}(\lambda_l)|^2. \tag{2.3}$$

Note that the three closest Fourier coefficients at frequencies λ_l were calculated at time points $t_0 - \mathrm{mod}(t_0) + l - m$, $t_0 - \mathrm{mod}(t_0) + l$ as well as $t_0 - \mathrm{mod}(t_0) + l + m$ and the above case distinction picks the one of the three closest to t_0, where in case of m even and equal distance between two the left one is chosen.

The moving local Fourier coefficients at time t_0 and all frequencies λ_l, $l = 1, \ldots, m$ incorporate the observations $X_{t_0 - \lfloor m/2 \rfloor - m}, \ldots, X_{t_0 + \lceil m/2 \rceil - 1 + m}$. Of those $3m$ observations, $X_{t_0 - 1 - \lfloor m/2 \rfloor}, \ldots, X_{t_0 + \lceil m/2 \rceil}$ occur in all of the moving local Fourier coefficients.

Remark 2.1. *Because* $l - \mathrm{mod}(t_0) \geqslant \lceil m/2 \rceil$ *is only possible for* $\mathrm{mod}(t_0) \leqslant \lfloor m/2 \rfloor$*, while* $l - \mathrm{mod}(t_0) < -\lfloor m/2 \rfloor$ *for* $\mathrm{mod}(t_0) > \lfloor m/2 \rfloor + 1$*, at most two of the cases in Definition 2.2 can occur for any given time point* t_0*. For* $\mathrm{mod}(t_0) = \lfloor m/2 \rfloor + 1$ *only the middle case occurs.*

Remark 2.2. *The assumption that we have data* $X_{-m+1,T}, \ldots, X_{T+m,T}$ *available can be abandoned by slightly changing the scheme of transformation, employing the ordinary Fourier transform for the first and last stretch and retrieving not one, but* m *Fourier coefficients.*

2.2 Moving inverse Fourier transform

In this section we define an inverse moving Fourier transform that is needed to obtain a locally stationary bootstrap sample in the time domain after having bootstrapped the moving Fourier coefficients in the frequency domain. This transform is not an inverse transform in an analytical sense as it does not yield the exact same sequence when applied to the moving Fourier transform of the original input sequence. However, it conserves statistical properties in an analogous way as the discrete inverse Fourier transform does for the TFT-bootstrap Kirch and Politis (2011).

To this end, we assume that we have a sequence $M\!F_1^*, \ldots, M\!F_T^*$, which will be the bootstrapped moving Fourier coefficients in this paper. Roughly speaking, we will then obtain moving local Fourier coefficients $M\!F_k^*(\lambda_l)$, $k = 1, \ldots, T$, $l = 1, \ldots, m$, from these in the same manner as in Definition 2.2. Finally, we use a discrete inverse Fourier transform of $M\!F_k^*(\cdot)$ at point t to obtain the bootstrap sample X_t^* at time t, $t = \lfloor m/2 \rfloor + 1, \ldots, T - \lceil m/2 \rceil + 1$.

Definition 2.3 (Moving inverse Fourier transform)**.**
Let $M\!F_1^*, \ldots, M\!F_T^*$ *be moving elements in the frequency domain with moving local counterparts* $M\!F_t^*(\lambda_j)$*,* $j = 1, \ldots, m$*, obtained analogously to Definition 2.2. The moving inverse Fourier transform is defined by*

$$\begin{aligned} X_t^* &:= \mathcal{F}^{-1}\left(M\!F_t^*(\lambda_1), M\!F_t^*(\lambda_2), \ldots, M\!F_t^*(\lambda_m); t\right) \\ &:= \frac{1}{\sqrt{2m+1}} \sum_{l=1}^{m} M\!F_t^*(\lambda_l)\, e^{i\lambda_l t} + \frac{1}{\sqrt{2m+1}} \sum_{l=1}^{m} \overline{M\!F_t^*(\lambda_l)}\, e^{-i\lambda_l t} \end{aligned} \tag{2.4}$$

with $\lambda_k = \frac{2\pi k}{N}$*,* $k = 0, \ldots, m$*, and* $t = \lfloor m/2 \rfloor + 1, \ldots, T - \lceil m/2 \rceil + 1$.

Remark 2.3. *Similarly, to the moving Fourier transform there are more coefficients* $M\!F_k^*$ *needed than are obtained back in the time domain. This is due to the shifting and similarly to Remark 2.2 can in practise be circumvented by using the discrete Fourier transformation for all* $t = 1, \ldots, \lfloor m/2 \rfloor + 1$ *for the first stretch and for all* $t = T - \lceil m/2 \rceil + 1, \ldots, T$ *for the last stretch.*

2.3 Locally stationary processes

Following Dahlhaus (1997, 2003) and Sergides (2008) we will use the following definition for locally stationary processes to obtain theoretic results.

Definition 2.4. *A sequence of stochastic processes* $\{X_{t,T},\ t = 1, \ldots, T\}$ *is called locally stationary if there exists a representation*

$$X_{t,T} = \sum_{j=-\infty}^{\infty} a_{t,T}(j)\varepsilon_{t-j}, \tag{2.5}$$

where the following holds

(a) $\varepsilon_t \overset{iid}{\sim} (0, 1)$ *with finite fourth moment* $E\varepsilon_t^4 < \infty$*,*

(b) $\sup_t |a_{t,T}(j)| \leqslant \frac{K}{l(j)}$*, and let* $\{l(j)\}$ *be a positive sequence with*

$$l(j) := \begin{cases} 1, & |j| \leqslant 1 \\ |j| \log^{1+\kappa} |j|, & |j| > 1 \end{cases}$$

for some $\kappa > 0$*.*

(c) There exist functions $a(\cdot, j) : (0, 1] \to \mathbb{R},\ j \in \mathbb{Z}$*, with*

(i) $\sup_t \left|a_{t,T}(j) - a(\frac{t}{T}, j)\right| \leqslant \frac{K}{Tl(j)}$*.*

(ii) $|a(u, j) - a(v, j)| \leqslant \frac{K|u-v|}{l(j)}$

(iii) $\sup_u \left|\frac{\partial^i a(u,j)}{\partial u^i}\right| \leqslant \frac{K}{l(j)}$*,* $i = 0, 1, 2, 3$*.*

Following Sergides (2008), Dahlhaus and Subba Rao (2006) and Subba Rao (2006) we define, for some $u \in (0, 1)$, the stationary process $\tilde{X}_t(u)$ by

$$\tilde{X}_t(u) := \sum_{j=-\infty}^{\infty} a(u, j)\varepsilon_{t-j}, \tag{2.6}$$

for which $X_{t,T} = \tilde{X}_t(u) + O_P\left(\left|\frac{t}{T} - u\right| + \frac{1}{T}\right)$ (cf. Sergides (2008), Equation (1.1.19), and Dahlhaus and Subba Rao (2006), p.4). We then define the time varying spectral density and covariances as the corresponding functions of the stationary approximation $\widetilde{X}_t(u)$, i.e. the time varying spectral density is given by

$$f(u, \lambda) = \frac{1}{2\pi}|A(u, \lambda)|^2, \tag{2.7}$$

with $A(u, \lambda) := \sum_{j=-\infty}^{\infty} a(u, j)e^{-i\lambda j}$, while the time varying covariance of lag h, $h \in \mathbb{Z}$ (at rescaled time u) is given by $c(u, h) := \int_{-\pi}^{\pi} f(u, \lambda)e^{i\lambda h}d\lambda$. Referring to Dahlhaus (2003), Equation (17), it holds $\text{cov}(X_{\lfloor uT \rfloor,T}, X_{\lfloor uT \rfloor+h,T}) = c(u, h) + O\left(\frac{1}{T}\right)$ uniformly in u and h.

Assumption A.1. *In the remainder of the paper, the time-varying covariances* $c(u, h)$ *are assumed to be absolutely summable for every* $u \in [0, 1]$*.*

2.4 Asymptotic statistical properties of the MLFT

In order to obtain mathematically precise asymptotic assertions in the context of locally stationary processes as above, we need the following assumptions on the sample size T and the segments length $2m+1$, which will be assumed throughout the remainder of the paper:

Assumption A.2.
- *$m \to \infty$ (for $T \to \infty$).*
- *$\frac{m^3}{T^2} = O(1)$ (for $T \to \infty$) i.e. the sample size increases considerably faster than the window size.*

We are now ready to state our first assertion showing that the expectation and more importantly variance of moving local Fourier coefficients behave analogously to corresponding quantities of Fourier coefficients of stationary time series:

Theorem 2.1. *Let $\{X_{t,T}\}$ be a locally stationary process as in Definition 2.4.*

(a) Then, it holds

$$
\begin{aligned}
&(i) && E\left(MF_{\lfloor uT \rfloor}(\lambda_l)\right) = 0, \qquad 0 \leqslant u \leqslant 1, \quad l = 1, \ldots, m,\\
&(ii) && \sup_{u\in[0,1]} \sup_{l=1,\ldots,m} |\operatorname{var}(MF_{\lfloor uT \rfloor}(\lambda_l)) - 2\pi f(u,\lambda_l)| \to 0 \qquad (T \to \infty).
\end{aligned}
$$

(b) If additionally $\sup_{u\in[0,1]} \sum_{j=-\infty}^{\infty} |a(u,j)|\sqrt{|j|} < \infty$, then

$$
\sup_{u\in[0,1]} \sup_{l=1,\ldots,m} |\operatorname{var}(MF_{\lfloor uT \rfloor}(\lambda_l)) - 2\pi f(u,\lambda_l)| = O\left(\frac{1}{\sqrt{m}}\right).
$$

In the classic stationary time series setting, Fourier coefficients are asymptotically uncorrelated. For the moving local Fourier coefficients this remains true as long as either the frequencies are far enough apart or if they are based on observations that are close enough together. Recall, that we calculate the frequencies $\lambda_1, \ldots, \lambda_m$ and then start over. Consequently, it can happen (see the case distinctions in the definition of the moving local Fourier coefficients), that for a given location $0 \leqslant u \leqslant 1$ the local Fourier coefficients (at $\lfloor uT \rfloor$) at frequencies λ_l and λ_k come from two stretches that are shifted by $m - |k-l|$ (rather than $|k-l|$) observations. If l and k are close but at the same time the Fourier coefficients come from such strongly shifted sets, the covariance can only be guaranteed to be asymptotically bounded but does not need to converge to zero.

The following theorem, characterizes this assertion in more details. Fortunately, for many application such as spectral density estimation the set of Fourier coefficients with non-vanishing correlation is sufficiently small to be asymptotically negligible (see Section 2.6):

Theorem 2.2. *Let $\{X_{t,T}\}$ be a locally stationary process as in Definition 2.4 and a_m be a sequence with $a_m \to \infty$, $\frac{a_m}{m} \to 0$.*

(a) It holds as $T \to \infty$

$$
\begin{aligned}
&(i) && \sup_{0\leqslant u\leqslant 1} \sup_{1\leqslant l,j\leqslant m} |\operatorname{cov}(MF_{\lfloor uT \rfloor}(\lambda_l), MF_{\lfloor uT \rfloor}(\lambda_j))| = O(1),\\
&(ii) && \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l,j\leqslant m\\ |l-j|\geqslant a_m}} \left|\operatorname{cov}(MF_{\lfloor uT \rfloor}(\lambda_l), MF_{\lfloor uT \rfloor}(\lambda_j))\right| = o(1),\\
&(iii) && \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l\neq j\leqslant m,\\ |l-j|\leqslant a_m, (l,j)\in\mathcal{M}(\lfloor uT \rfloor, m)}} \left|\operatorname{cov}(MF_{\lfloor uT \rfloor}(\lambda_l), MF_{\lfloor uT \rfloor}(\lambda_j))\right| = o(1),
\end{aligned}
$$

where the set $\mathcal{M}(k,m)$ contains all pairs (l,j), which are in the same case of Definition 2.2 for given k, i.e.

$$\mathcal{M}(k,m) = \{(l,j) \in \{1,\dots,m\}^2 : l,j \in [\mathrm{mod}(k) - \lfloor m/2 \rfloor, \mathrm{mod}(k) + \lceil m/2 \rceil)\} \\ \cup \{(l,j) \in \{1,\dots,m\}^2 : l,j \notin [\mathrm{mod}(k) - \lfloor m/2 \rfloor, \mathrm{mod}(k) + \lceil m/2 \rceil)\}.$$

(b) If additionally $\sup_{u\in[0,1]} \sum_{j=-\infty}^{\infty} |a(u,j)|\sqrt{|j|} < \infty$, *then*

$$(i) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l,j\leqslant m \\ |l-j|\geqslant a_m}} \left|\mathrm{cov}(M\!F_{\lfloor uT\rfloor}(\lambda_l), M\!F_{\lfloor uT\rfloor}(\lambda_j))\right| = O\left(\frac{1}{a_m} + \frac{1}{\sqrt{m}}\right),$$

$$(ii) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l\neq j\leqslant m, \\ |l-j|\leqslant a_m, (l,j)\in\mathcal{M}(\lfloor uT\rfloor,m)}} \left|\mathrm{cov}(M\!F_{\lfloor uT\rfloor}(\lambda_l), M\!F_{\lfloor uT\rfloor}(\lambda_j))\right| = O\left(\frac{a_m}{m} + \frac{1}{\sqrt{m}}\right).$$

Remark 2.4. *Similar restrictions also apply to the moving Fourier coefficients $M\!F_l, M\!F_j$ with $|l-j| < 3m$, i.e. they are asympotically uncorrelated if either the corresponding frequencies are far apart ($|\mathrm{mod}(l) - \mathrm{mod}(j)| \geqslant a_m$) or if they are calculated from the same stretch of frequencies with no jump from frequency λ_m to λ_1 in between ($|\mathrm{mod}(l) - \mathrm{mod}(j)| = |l-j|$); see Lindner (2014), Theorem 5.4. As soon as $|l-j| \geqslant 3m$, all moving Fourier coefficients become asymptotically uncorrelated due to the fact that the underlying observations grow further and further apart (see Lemma 5.1 in Lindner (2014)).*

Similarly, one can obtain the following result for the covariance of the moving local periodogram:

Theorem 2.3. *Let $\{X_{t,T}\}$ be a locally stationary process as in Definition 2.4 and a_m be a sequence with $a_m \to \infty$, $\frac{a_m}{m} \to 0$.*

(a) It holds for any $0 \leqslant u \leqslant 1$ as $T \to \infty$

$$(i) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{1\leqslant l,j\leqslant m} |\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))| = O(1),$$

$$(ii) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l,j\leqslant m \\ |l-j|\geqslant a_m}} \left|\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))\right| = o(1),$$

$$(iii) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l\neq j\leqslant m, \\ |l-j|\leqslant a_m, (l,j)\in\mathcal{M}(\lfloor uT\rfloor,m)}} \left|\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))\right| = o(1),$$

where the set $\mathcal{M}(\cdot,m)$ is defined as in Theorem 2.2.

(b) If additionally $\sup_{u\in[0,1]} \sum_{j=-\infty}^{\infty} |a(u,j)|\sqrt{|j|} < \infty$, *then*

$$(i) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l,j\leqslant m \\ |l-j|\geqslant a_m}} \left|\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))\right| = O\left(\frac{1}{a_m^2} + \frac{1}{\sqrt{m}}\right),$$

$$(ii) \qquad \sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l\neq j\leqslant m, \\ |l-j|\leqslant a_m, (l,j)\in\mathcal{M}(\lfloor uT\rfloor,m)}} \left|\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))\right| = O\left(\frac{a_m^2}{m^2} + \frac{1}{\sqrt{m}}\right).$$

In particular, one gets

$$\sup_{0\leqslant u\leqslant 1} \sup_{\substack{1\leqslant l\neq j\leqslant m, \\ (l,j)\in\mathcal{M}(\lfloor uT\rfloor,m)}} \left|\mathrm{cov}(M\!I_{\lfloor uT\rfloor}(\lambda_l), M\!I_{\lfloor uT\rfloor}(\lambda_j))\right| = O\left(\frac{1}{\sqrt{m}}\right).$$

2.5 A useful tool: Approximation by the MLFT of independent random variables

To obtain the results of the previous section we made explicit use of the linear structure of locally stationary processes. Similarly to classic stationary linear processes (see e.g. Theorem 10.3.1 in Brockwell and Davis (2006)) we used the following relationship between the (local) moving Fourier coefficients of the locally stationary process as well as the (local) moving Fourier coefficients of the corresponding i.i.d. sequence.

Theorem 2.4. *Let $\{X_{t,T}\}$ be a locally stationary process as in Definition 2.4 in addition to $A(u,\lambda)=\sum_{j=-\infty}^{\infty} a(u,j)e^{-i\lambda j}$. Then, the following decomposition holds*

$$MF_{\lfloor uT\rfloor}(\lambda_l) = A(u,\lambda_l) MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l) + R_{\lfloor uT\rfloor,m}(\lambda_l),$$

where the following assertions hold for the remainder term $R_{\lfloor uT\rfloor,m}(\lambda_l)$:

(a)o It holdsi

$$\begin{aligned} &(i) && \mathrm{E}\, R_{\lfloor uT\rfloor,m}(\lambda_l) = 0,\\ &(ii) && \sup_{u\in[0,1]}\sup_{l=1,\ldots,m} \mathrm{E}\,|R_{\lfloor uT\rfloor,m}(\lambda_l)|^4 \to 0 \text{ as } m\to\infty. \end{aligned}$$

(b)o If additionally $\sup_{u\in[0,1]}\sum_{j=-\infty}^{\infty}|a(u,j)|\sqrt{|j|}<\infty$, then we even get

$$\sup_{u\in[0,1]}\sup_{l=1,\ldots,m} \mathrm{E}\,|R_{\lfloor uT\rfloor,m}(\lambda_l)|^4 = O\left(\frac{1}{m^2}\right).$$

Similarly, we obtain the following result for moving local periodograms.

Corollary 2.1. *(a) Let $\{X_{t,T}\}$ be a locally stationary process as in Definition 2.4 with time varying spectral density $f(u,\lambda)$. Then, the following decomposition holds*

$$MI_{\lfloor uT\rfloor,m}(\lambda_l) = 2\pi f(u,\lambda_l)\; MI^{\varepsilon}_{\lfloor uT\rfloor,m}(\lambda_l) + R'_{\lfloor uT\rfloor,m}(\lambda_l), \tag{2.8}$$

with $\sup_{u\in[0,1]}\sup_{l=1,\ldots,m}\mathrm{E}\,|R'_{\lfloor uT\rfloor,m}(\lambda_l)|^2\to 0$ as $T\to\infty$.

(b) If additionally $\sup_{u\in[0,1]}\sum_{j=-\infty}^{\infty}|a(u,j)|\sqrt{|j|}<\infty$, then we even get

$$\sup_{u\in[0,1]}\sup_{l=1,\ldots,m} \mathrm{E}\,|R'_{\lfloor uT\rfloor,m}(\lambda_l)|^2 = O\left(\frac{1}{m}\right).$$

2.6 Applications to spectral density estimation

Before coming to the main result for spectral density estimators we give a kind of law of large numbers for the moving local periodograms that is needed in the proof but may also be of independent interest in particular as it duplicates a corresponding result for stationary time series:

Theorem 2.5. *Let $X_{t,T}$ be a locally stationary process as in Definition 2.4 with spectral density $f(u,\lambda_j)$ that is uniformly bounded away from 0, then it holds for $0\leqslant u\leqslant 1$*

$$\frac{1}{2m+1}\sum_{j=1}^{2m}\frac{MI_{\lfloor uT\rfloor,m}(\lambda_j)}{2\pi f(u,\lambda_j)} \xrightarrow{P} 1.$$

Instead of considering the (local) covariance structure of a time series one can also consider the (local) spectral density estimation. While (moving local) periodograms are asymptotically unbiased (which follows essentially from Theorem 2.1), they are not consistent. As for stationary time series we will therefore smooth the periodogram, i.e. consider kernel density estimators based on periodograms. In the context of locally stationary time series, Sergides (2008) considered a corresponding kernel density estimator based on the local periodogram (where for a given point in time $0 \leqslant u \leqslant 1$ all frequencies are calculated based on the observations in a local environment).

In this section, we investigate the consistency of the corresponding local spectral density estimator based on the moving local periodogram.

To elaborate, the smoothed moving periodogram $\hat{f} : \{1, \ldots, T\} \to \mathbb{R}$ is defined by

$$\hat{f}(k) := \frac{1}{2m+1} \sum_{t=-m}^{2m} K_h\left(\lambda_{mod(k)} - \lambda_t\right) MI_k\left(\lambda_t\right), \tag{2.9}$$

with $MI_{k,m}(\lambda_t)$ being the moving local periodogram as in Definition 2.2 and $K_h(\cdot) := \frac{1}{h} K\left(\frac{\cdot}{h}\right)$ for a kernel K and a bandwidth h fulfilling:

Assumption A.3. *(i) K is a nonnegative, symmetric function with compact support.*

(ii) $\int K(x)dx = 1,\ |K(x)| \leqslant const.,$

$$\frac{2\pi}{(2m+1)h} \sum_{j \in \mathbb{Z}} K\left(\frac{2\pi j}{(2m+1)h}\right) = \int K(x)dx + o(1) = 1 + o(1).$$

(iii) K is uniformly Lipschitz continuous.

(iv) $h \to 0$ $(T \to \infty)$ and $hm^{\frac{1}{4}} \to \infty$.

The following Theorem shows that the smoothed periodogram is in fact a consistent (even in some uniform sense) estimator for the local spectral density.

Theorem 2.6. *Let $\{X_{t,T}\}$ be a locally stationary time series as in Definition 2.4 with time varying spectral density f bounded away from 0 and fulfilling a uniform Lipschitz condition in the second argument (for any $0 \leqslant u \leqslant 1$). Furthermore, let Assumptions A.3 be fulfilled. Additionally, let $\sup_{u\in[0,1]} \sum_{j=-\infty}^{\infty} |a(u,j)|\sqrt{|j|} < \infty$. Then, for every $u \in [0,1]$, the estimator $\hat{f}$ as in (2.9) fulfills*

$$\sup_{k \in \{\lfloor uT \rfloor - \lceil \frac{m}{2} \rceil + 1, \ldots, \lfloor uT \rfloor + \lfloor \frac{m}{2} \rfloor\}} \left| \hat{f}(k) - f\left(u, \lambda_{mod(k)}\right) \right| = o_P(1).$$

Remark 2.5. *The bandwidth condition is slightly stronger than in the stationary case. This is due to the rate $O(1/\sqrt{m})$ in Theorem 2.3, which is obtained easily from the approximations in Section 2.5. In the stationary case, one can get the stronger rate $O(1/m)$ by much more involved arguments (see Brockwell and Davis (2006), proof of Theorem 10.3.2). If one can improve this rate for the moving local periodogram as well, then the bandwidth condition $hm^{\frac{1}{3}} \to \infty$ is sufficient for Theorem 2.6, where in the proof $\alpha_m = h/m^{1/3}$ needs to be chosen.*

3 Applications to the Bootstrap

In the previous sections we have seen that the moving Fourier coefficients of locally stationary time series are in some sense (almost) asymptotically uncorrelated similar to global Fourier coefficients of stationary time series. This is in contrast to the sets of all local periodograms at all frequencies which is not independent due to the duplicated information. It is this dependence between those periodogram ordinates that make it difficult or even impossible to use the local periodogram in a bootstrap because such a bootstrap needed to mimick the full dependence structure correctly. On the other hand, our moving scheme is very well suited for such a task as it manages to get rid of this dependence by thinning. Nevertheless, local periodograms have already been used in the context of bootstrapping by Sergides (2008) but only pointwise to obtain information about local quantities such as the autocorrelation at a given point in rescaled time.

In this section, we introduce moving versions of three important classes of frequency domain bootstrapping, where we will prove that the corresponding bootstrap replicates in the time domain correctly mimic the local covariance structure for the simplest bootstrap (the wild TFT-bootstrap). We will then compare the performance of all three schemes using the example of uniform confidence bands for the local autocorrelation at lag 1, for which such a moving scheme is essential. Finally, the method is applied to some wind data.

3.1 Moving wild TFT bootstrap

The following bootstrap is a generalization of the (wild) TFT bootstrap as proposed by Kirch and Politis (2011) to the new scheme of moving Fourier coefficients.

Step 1: The observed time series is transformed using the moving Fourier coefficients as in Definition 2.1 (for practical purposes in addition to Remark 2.2) resulting in $M\!F_1, \ldots, M\!F_T$.

Step 2: Let G_k, G_{k+T}, $k = 1, \ldots, T$, be independent identically standard normal random variables (independent of the original time series). Generate the bootstrap samples $M\!F_k^* = x_k^* + iy_k^*$ according to

$$x_k^* := \sqrt{\pi \hat{f}\left(\frac{k}{T}, \lambda_{mod(k)}\right)} G_k, \qquad y_k^* := \sqrt{\pi \hat{f}\left(\frac{k}{T}, \lambda_{mod(k)}\right)} G_{k+T},$$

where $\hat{f}\left(\frac{k}{T}, \lambda_{mod(k)}\right)$ is an estimator (based on the moving Fourier transform) that fulfills

$$\max_{k \in \left\{\lfloor uT \rfloor - \lceil \frac{m}{2} \rceil + 1, \ldots, \lfloor uT \rfloor + \lfloor \frac{m}{2} \rfloor\right\}} \left| \hat{f}\left(\frac{k}{T}, \lambda_{mod(k)}\right) - f\left(u, \lambda_{mod(k)}\right) \right| = o_P(1).$$

The square of these bootstrap Fourier coefficients give a bootstrap version of the periodogram for periodogram statistics. For other statistics, we need to backtransform these sequence into the time domain as in the following step.

Step 3: The moving bootstrap coefficients are transformed back using a moving version of the inverse Fourier transform as in Definition 2.3 (in addition to Remark 2.3).

$$\begin{aligned} X_t^* &= \mathcal{F}^{-1}\left(M\!F_t^*(\lambda_1), M\!F_t^*(\lambda_2), \ldots, M\!F_t^*(\lambda_m); t\right) \\ &= \frac{1}{\sqrt{2m+1}} \sum_{l=1}^{m} M\!F_t^*(\lambda_l)\, e^{i\lambda_l t} + \frac{1}{\sqrt{2m+1}} \sum_{l=1}^{m} \overline{M\!F_t^*(\lambda_l)}\, e^{-i\lambda_l t} \end{aligned}$$

with $\lambda_k := \frac{2\pi k}{N}$, $k = 0, \ldots, m$, denoting the Fourier frequencies and $t = 1, \ldots, T$ and $MF_t^*(\lambda_j)$, $j = 1, \ldots, m$, obtained analogously to Definition 2.2 from $MF_1^*, MF_2^*, \ldots$.

This finally yields a bootstrap replicate $X_{1,T}^*, X_{2,T}^*, \ldots, X_{T,T}^*$ of the original time series in the time domain.

The above bootstrap uses a moving version of the Whittle likelihood as approximation to the true dependence structure in the frequency domain.

The following theorem proves that the above bootstrap correctly mimics the local covariance structure of the original locally stationary time series:

Theorem 3.1. *Let the assumptions of Theorem 2.6 be fulfilled. Then, it holds*

$$\sup_{|h| \leqslant m} \left| Cov^*(X^*_{\lfloor uT \rfloor, T}, X^*_{\lfloor uT \rfloor + h, T}) - c(u,h) \right| = o_P(1).$$

Remark 3.1. *If $h > m$, then $X^*_{t,T}$ and $X^*_{t+h,T}$ are independent due to the m-dependence of the bootstrap scheme. This is consistent with the original covariance structure in the sense that by Dahlhaus and Polonik (2009), Equation (51), it holds as $h \to \infty$*

$$\sup_t \left| c_T\left(\frac{t}{T}, h\right) \right| \leqslant \sum_{j=-\infty}^{\infty} \frac{K}{l(j)l(j+h)} = O\left(\frac{1}{l(h)}\right) = o(1).$$

Instead of using the wild bootstrap in the frequency domain, one can also apply other frequency domain bootstrap methods in Step 2 as proposed by Kirch and Politis (2011) in the stationary case such as the residual-based or local bootstrap. For details we refer to Lindner (2014) Section 7.1.

Remark 3.2. *The assertion of the theorem remains true for other frequency domain bootstrap schemes as long as they fulfill the following assumptions:*

(i) $E^*(x_k^*) = E^*(y_k^*) = 0, \quad \forall k = 1, \ldots, T.$

(ii) Independence of $\{x_k^*, y_l^*, k, l = 1, \ldots, m, k \neq l\}$.

(iii)

$$\sup_{k \in \{\lfloor uT \rfloor - \lceil \frac{m}{2} \rceil + 1, \ldots, \lfloor uT \rfloor + \lfloor \frac{m}{2} \rfloor\}} |\mathrm{var}(x_k^*) - \pi f(u, \lambda_k)| = o_P(1).$$

$$\sup_{k \in \{\lfloor uT \rfloor - \lceil \frac{m}{2} \rceil + 1, \ldots, \lfloor uT \rfloor + \lfloor \frac{m}{2} \rfloor\}} |\mathrm{var}(y_k^*) - \pi f(u, \lambda_k)| = o_P(1).$$

Assumptions (i) and (ii) are fulfilled by construction for the wild as well as the local or residual-based bootstrap. For the wild bootstrap Assumption (iii) is fulfilled due to Theorem 2.6.

3.2 Alternative moving bootstrap methods in the frequency domain

Because first-generation frequency domain bootstrap methods as introduced in the previous section can only mimic the first and second order of time series correctly, several second-generation frequency domain bootstrap methods have been introduced in the literature. These hybrid methods also correctly mimic the fourth order structure of the time series to a certain extend thus extending the class of statistics or the class of time series for which the bootstrap is valid. This is obtained by combining a bootstrap in the time domain with a non-parametric correction of the underlying spectral density in the frequency domain.

3.2.1 Moving autoregressive aided periodogram bootstrap

For stationary time series Kreiss and Paparoditis (2003) propose the autoregressive aided periodogram bootstrap which combines a parametric autoregressive bootstrap with a non-parametric correction of the bootstrap periodogram in the frequency domain. This bootstrap has been considered in a locally stationary context by Sergides (2008) but only for the local Fourier transform so that it can correctly mimic the local structure of the time series and be applied to local quantities of interest but not global ones.

However, using the moving Fourier transform introduced in this paper, we can now give a global adaptation of this method.

Step 1: For every point in time $m \leqslant t \leqslant T-m$ we fit an autoregressive model of order 1 to the data $X_{t-m,T}, \ldots, X_{t+m,T}$ and calculate the local Yule-Walker-estimators of the parameters $\hat{a}^{(t)}$ as well as $\hat{\sigma}^{(t)}$. The beginning and end of the time series can be dealt with similarly as in Remark 2.2.

From these, we calculate the centered rescaled residuals $\hat{\varepsilon}_{t,T} := \tilde{\varepsilon}_{t,T} - \frac{1}{T}\sum_{\tau=1}^{T} \tilde{\varepsilon}_{\tau,T}$, where

$$\tilde{\varepsilon}_{t,T} := \frac{1}{\hat{\sigma}^{(t)}}\left(X_{t,T} - \hat{a}^{(t)} X_{t-1,T}\right), \qquad t = 2, \ldots, T.$$

Step 2: We now obtain a bootstrap sample $\varepsilon^{+}_{t,T}$, $t = 1, \ldots, T$, by a local Efron bootstrap from $\hat{\varepsilon}_{j,T}$, $j = t-m, \ldots, t+m$. From this we obtain the time domain bootstrap observations $X^{+}_{1,T} := X_{1,T}$ and

$$X^{+}_{t,T} := \hat{a}^{(t)} X^{+}_{t-1,T} + \hat{\sigma}^{(t)} \varepsilon^{+}_{t,T},\ t = 2, \ldots, T.$$

Step 3: As in Definition 2.2 in addition to Remark 2.2 we calculate the moving local coefficients $M\!F^{+}_{t}(\lambda_1), \ldots, M\!F^{+}_{t}(\lambda_m)$ of $\{X^{+}_{t,T}\}$ as well as the moving periodogram $M\!I_t(\lambda_j)$. The bootstrap periodograms are then obtained by

$$M\!I^{*}_{t}(\lambda_j) := \hat{q}\left(\frac{t}{T}, \lambda_j\right) \cdot M\!I^{+}_{t}(\lambda_j),$$

where

$$\hat{q}\left(\frac{t}{T}, \lambda\right) := \frac{1}{2m+1} \sum_{j=-m}^{2m} K_h(\lambda - \lambda_j) \frac{M\!I^{+}_{t,m}(\lambda_j)}{\hat{f}^{(t)}_{AR}\left(\frac{t}{T}, \lambda_j\right)},$$

$$\hat{f}^{(t)}_{AR}\left(\frac{t}{T}, \lambda_j\right) := \frac{(\hat{\sigma}^{(t)})^2}{2\pi} \cdot \frac{1}{\left|1 - \hat{a}^{(t)} e^{-i\lambda_j}\right|^2}.$$

The algorithm as described above produces local bootstrap periodograms and as such can be used in situations where the statistical quantities of interest are based on these local periodograms. In all other situations a similar correction of the local Fourier coefficients in addition to the inverse Fourier transform as in Definition 2.3 can be used (for the stationary case we refer to Jentsch and Kreiss (2010)).

3.2.2 Moving hybrid wild bootstrap

For stationary time series Kreiss and Paparoditis (2012b) propose a hybrid wild bootstrap that combines a three-point wild bootstrap in the time domain with a non-parametric correction of the periodogram in the frequency domain. The three-point distribution is thereby chosen in such a way that it captures fourth-order properties of the underlying time series. This method has already been extended to the locally stationary case by the same authors Kreiss and Paparoditis (2012a) but their adaptation is based on a global Fourier transform and has to be custom-made for the statistic of interest.

The following algorithm uses the same idea but based on the moving local Fourier transform as proposed in this paper:

Step 1: Generate a sample $\varepsilon_1^*, \ldots, \varepsilon_T^*$ of length T of iid random variables meeting

$$P\left(\varepsilon_t^* = \sqrt{\tilde{\kappa}_4^t}\right) = P\left(\varepsilon_t^* = -\sqrt{\tilde{\kappa}_4^t}\right) = \frac{1}{2\tilde{\kappa}_4^t},$$
$$P(\varepsilon_t^* = 0) = 1 - \frac{1}{\tilde{\kappa}_4^t},$$

where $\tilde{\kappa}_4^t := \hat{\kappa}_4\left(\frac{t}{T}\right) + 3$ with

$$\hat{\kappa}_4\left(\frac{k}{T}\right) := \frac{\hat{G}_1\left(\frac{k}{T}\right) - \hat{G}_2\left(\frac{k}{T}\right)}{\hat{G}_3\left(\frac{k}{T}\right)},$$

$$\text{where} \quad \hat{G}_1\left(\frac{k}{T}\right) := \sum_{j=-m}^{2m} K_h(0-\lambda_j)\cdot\ MI_{(2),k}(\lambda_j),$$

$$\hat{G}_2\left(\frac{k}{T}\right) := \sum_{l=0}^{2m} \left(MI_k(\lambda_l)\right)^2, \qquad \hat{G}_3\left(\frac{k}{T}\right) := \left(\sum_{l=0}^{2m} MI_k(\lambda_l)\right)^2.$$

Here $MI_k(\lambda)$ denotes the moving local periodogram as in Definition 2.2 in addition to Remark 2.2 while $MI_{(2),k}(\lambda)$ denotes the moving local periodogram of the squared time series $\{X_{t,T}^2\}$.

Step 2: Calculate the moving local Fourier coefficients $\{MF_t^{\varepsilon^*}(\lambda_j)\}$ of the bootstrap errors $\{\varepsilon_1^*\}$ as in Definition 2.2 in addition to Remark 2.2.

Step 3: Generate the bootstrap observations in the time domain by

$$X_{t,T}^* := \frac{1}{\sqrt{2m+1}} \sum_{j=0}^{m} \sqrt{\hat{f}\left(\frac{t}{T}, \lambda_j\right)} \left(MF_t^{\varepsilon^*}(\lambda_j)e^{it\lambda_j} + \overline{MF_t^{\varepsilon^*}(\lambda_j)}e^{-it\lambda_j}\right),$$

where $\hat{f}$ is an estimator of the spectral density, fulfilling

$$\max_{k\in\{\lfloor uT\rfloor - \lceil \frac{m}{2}\rceil + 1, \ldots, \lfloor uT\rfloor + \lfloor \frac{m}{2}\rfloor\}} \left|\hat{f}\left(\frac{k}{T}, \lambda_{mod(k)}\right) - f\left(u, \lambda_{mod(k)}\right)\right| = o_P(1).$$

4 Simulation study and data analysis

We will now illustrate the potential of the moving local Fourier transform in addition to corresponding bootstrap methods by generating simultaneous (uniform) confidence bands for the local autocorrelations of order 1. Based on the above bootstrap methods one can construct simultaneous confidence bounds according to the following algorithm (confer also Neumann and Polzehl (1998), Sun and Loader (1994) or Lenhoff et al. (1999)).

Step 1: For every $u \in L[0,1] = \{t/T, t = 1, \ldots, T\}$ calculate an estimate $\hat{\rho}(u,1)$ of the autocorrelation function based on the moving local periodogram.

	mTFT	mAAPB	mH
DGP 1a	0.96	0.99	1.00
DGP 1b	0.95	0.99	1.00
DGP 1c	0.97	0.99	1.00

Table 4.1: Empirical coverage probability for variable width confidence bounds at $\alpha = 0.05$

Step 2: Generate B bootstrap time series by employing a moving bootstrap. For each time series $\left\{X_{t,T}^{*,b}\right\}$, $b = 1, \ldots, B$, estimate the autocorrelation $\hat{\rho}^b(u,1)$ for every $u \in L[0,1]$ as well as the standard deviation $\hat{\sigma}_{\hat{\rho}}(u)$ of $\hat{\rho}$.

Step 3: Choose $C_{boot} > 0$ such that

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\left\{\max_{u \in L[0,1]} \frac{\left|\hat{\rho}^b(u,1) - \hat{\rho}(u,1)\right|}{\hat{\sigma}_\rho(u)} \leqslant C_{boot}\right\} \geqslant 1 - \alpha,$$

for some prescribed α, $0 < \alpha < 1$.

Alternatively, choose $C'_{boot} > 0$ such that

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\left\{\max_{u \in L[0,1]} \left\{\left|\hat{\rho}^b(u,1) - \hat{\rho}(u,1)\right|\right\} \leqslant C'_{boot}\right\} \geqslant 1 - \alpha,$$

for some prescribed α, $0 < \alpha < 1$.

Step 4: The simultaneous $1 - \alpha$ variable width confidence band for $\rho(u,1)$, $0 \leqslant u \leqslant 1$, is then given by

$$CB_{variable} := [\hat{\rho}(u,1) - C_{boot} \cdot \hat{\sigma}_\rho(u),\ \hat{\rho}(u,1) + C_{boot} \cdot \hat{\sigma}_\rho(u)]$$

and the simultaneous $1 - \alpha$ fixed width confidence band by

$$CB_{fixed} := [\hat{\rho}(u,1) - C'_{boot},\ \hat{\rho}(u,1) + C'_{boot}].$$

In our simulations, the moving bootstrap method, as referred to in Step 2, will either be the moving TFT-bootstrap, the moving autoregressive aided periodogram bootstrap or the moving wild hybrid bootstrap. We will use $B = 500$ bootstrap replicates, a bandwidth of $m = 100$ for the moving transform and empirical results will be based on $R = 200$ repetitions, $\alpha = 0.05$.

We will use the following time-varying AR(1)-process

$$X_{t,T} = a_{t,T} \cdot X_{t-1,T} + \varepsilon_t, \quad a_{t,T} = \left(1 - \frac{t}{T}\right) \cdot (-0.6) + \frac{t}{T} \cdot 0.6. \qquad \text{(DGP 1)}$$

as well as the following time-varying MA(1)-process

$$X_{t,T} = 1.1 \cdot \cos\left(1.5 - \cos\left(\frac{4\pi\, i}{T}\right)\right) \cdot \varepsilon_{t-1} + \varepsilon_t, \qquad \text{(DGP 2)}$$

in simulations, where we use i.i.d. (a) standard normal, (b) standardized exponential and (c) standardized χ^2 random variables. Figure 4.1 displays two such realizations.

To account for the boundary effects which occur as we don't use the moving versions of the bootstrap for the very first and the very last observations, we only evaluate the

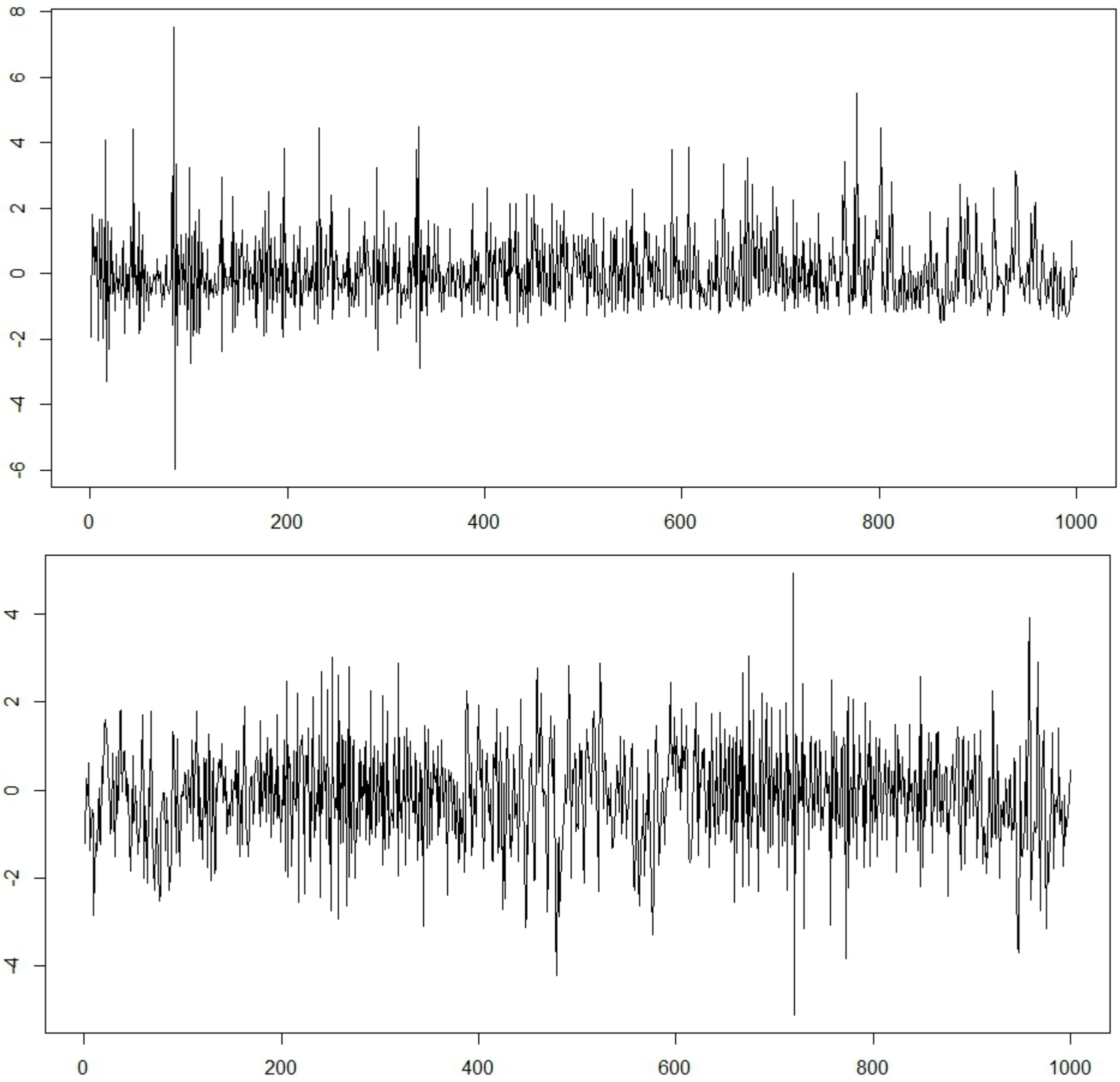

Figure 4.1: A realization of DGP 1c (top) and DGP 2a (bottom)

simulations in between $t = 200$ and $t = 800$. The below figures, however, display the whole range of $t = 1$ to $t = 1000$. One can clearly see – for example in Figure 4.2 – the effect of the blockwise bootstrap in the beginning and at the end.

Table 4.1 and Figure 4.2 show the results for the variable width confidence bounds and DGP 1. In this situation the local TFT-bootstrap has a coverage probability closest to the target value while the local autoregressive aided periodogram bootstrap as well as the local hybrid bootstrap are rather conservative, i.e. the confidence bounds are too wide, although this may partly be due to the fact that we only evaluate in the middle of the time series.

The results for DGP 2 as given in Table 4.2 and Figure 4.3 look quite different, where the coverage probability is too low in all cases. Only the moving autoregressive aided periodogram bootstrap processes somewhat acceptable results. However, when looking at Figure 4.3 it becomes apparent, where the problem lies as all bootstrap methods struggle at the locations where the autocorrelation structure changes relatively quickly (as compared to the global bandwidth m). This is an effect that is intrinsic to all such methods with fixed global bandwidth choice m. In fact, it is rather suprising that the

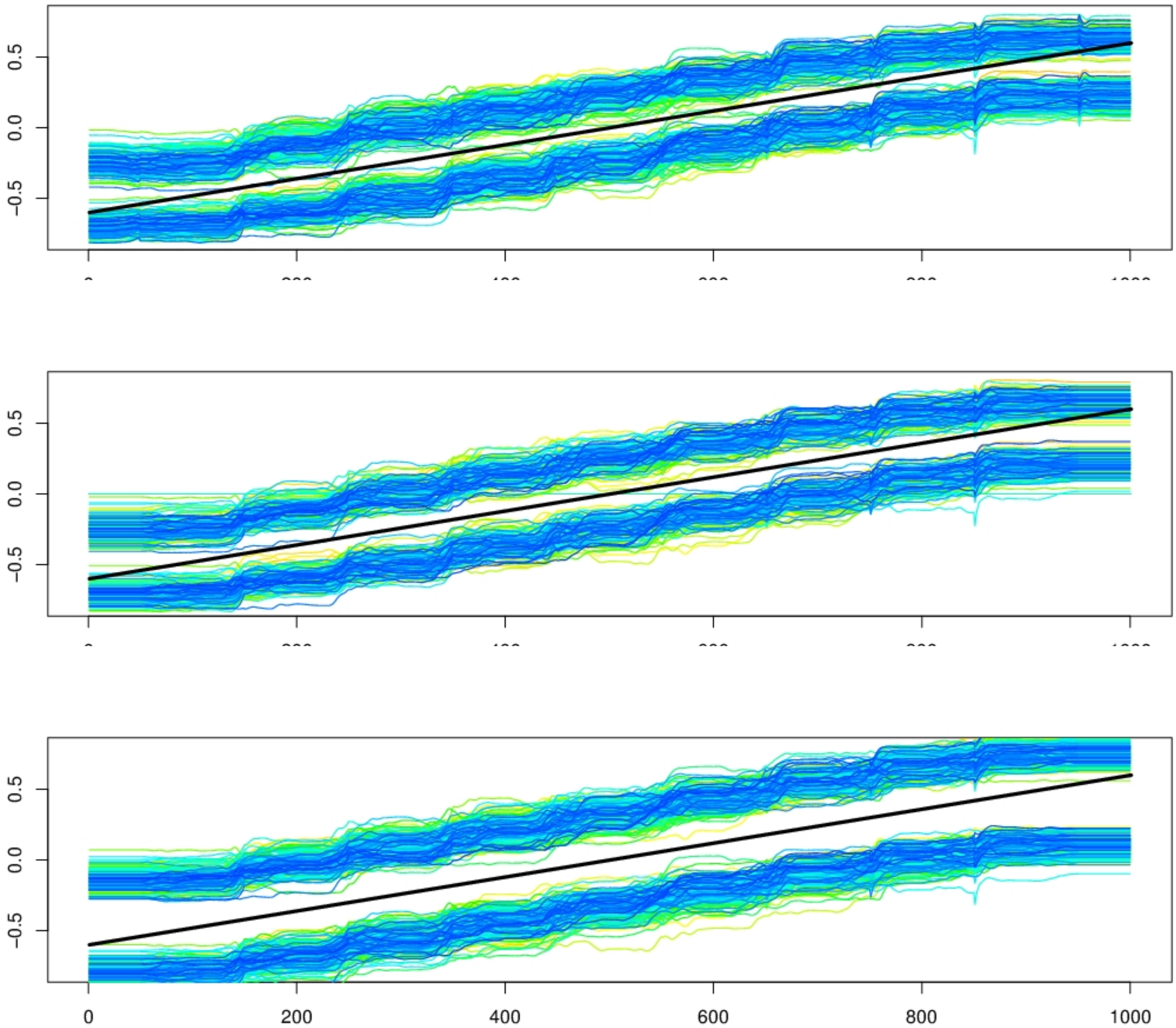

Figure 4.2: Confidence bands (variable width) for the time-varying autocorrelation at lag 1 of moving version of (a) TFT bootstrap, (b) AAP bootstrap and (c) wild hybrid bootstrap for DGP1a for different realizations

	mTFT	mAAPB	mH
DGP 2a	0.02	0.86	0.41
DGP 2b	0.01	0.90	0.23
DGP 2c	0.02	0.91	0.32

Table 4.2: Empirical coverage probability for variable width confidence bounds at $\alpha = 0.05$

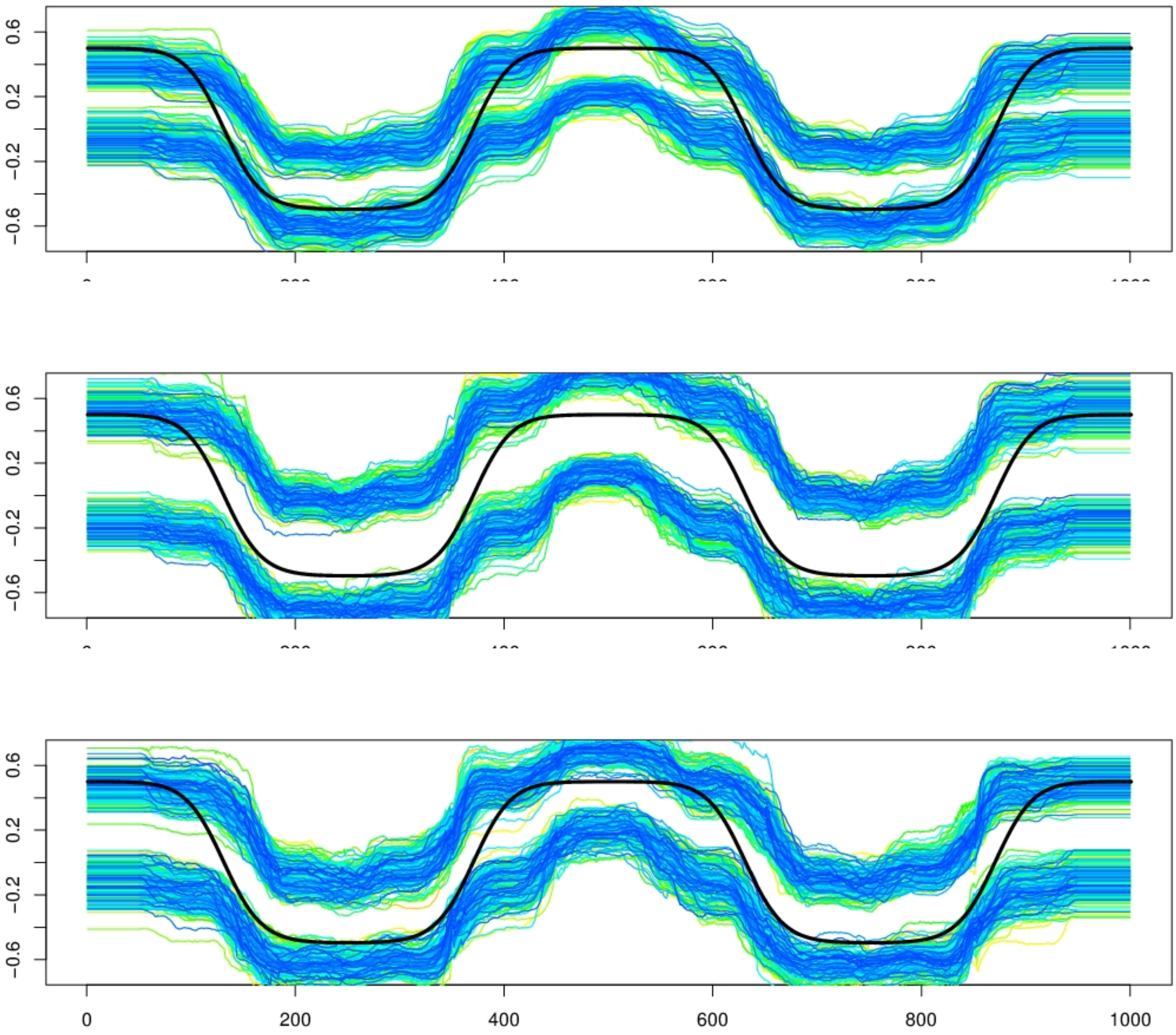

Figure 4.3: Confidence bands (variable width) for the time-varying autocorrelation at lag 1 of moving version of (a) TFT bootstrap (fixed width), (b) AAP bootstrap and (c) wild hybrid bootstrap for DGP2a for different realizations

moving autoregressive aided periodogram bootstrap works so well considering that the underyling local bootstrap in the time domain is based on a time-varying autoregressive approximation, while the true nature is a time-varying moving-average process.

We will now apply this methodology to a data set containing hourly data on the wind speed at the Meteorological Office station in Aberporth, Wales. This data set has already been studied by Hunt and Nason (2001), Nason and Sapatinas (2002), Eckley and Nason (2014). In order to remove the trend, we use differencing (see also Eckley and Nason (2014)). The resulting time series, as well as its estimated time-varying spectral density are displayed in Figure 4.4.

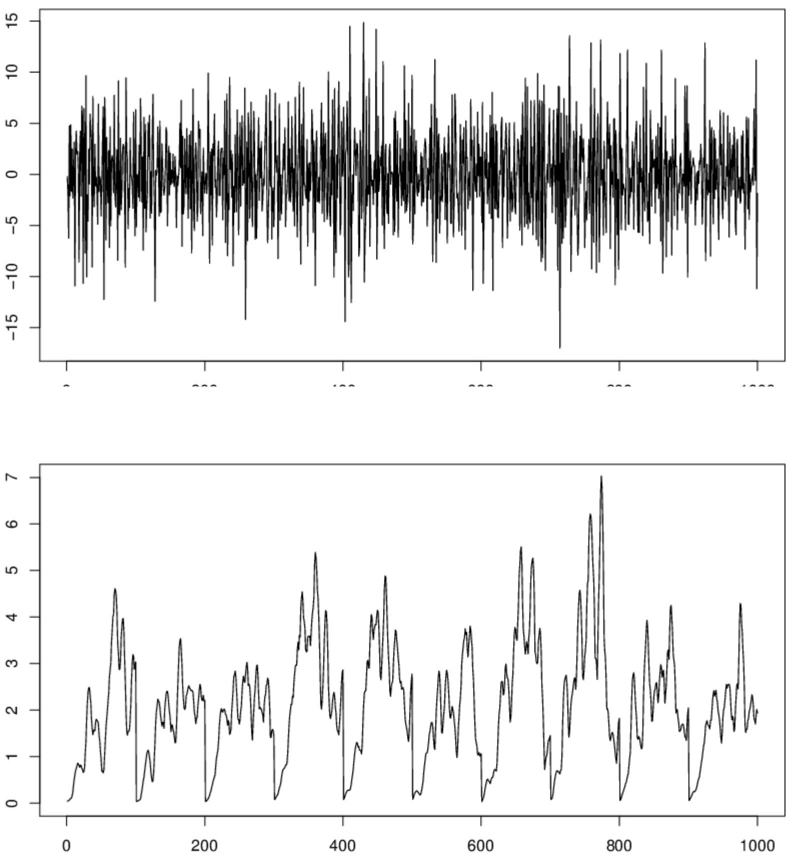

Figure 4.4: Time series ΔX_t of the wind data (top) and estimated time-varying spectral density (bottom).

Figure 4.5 displays the estimated time-varying autocorrelation function at lag 1 of the data. Using the mTFT-bootstrap, a 95%-confidence band of fixed width has been caluculated (black lines). Since it is possible to fit a horizontal line within the uniform confidence bounds it cannot be concluded that the autocorrelation at lag one of the data set is in fact time-varying – even though the local estimate does show some time variation.

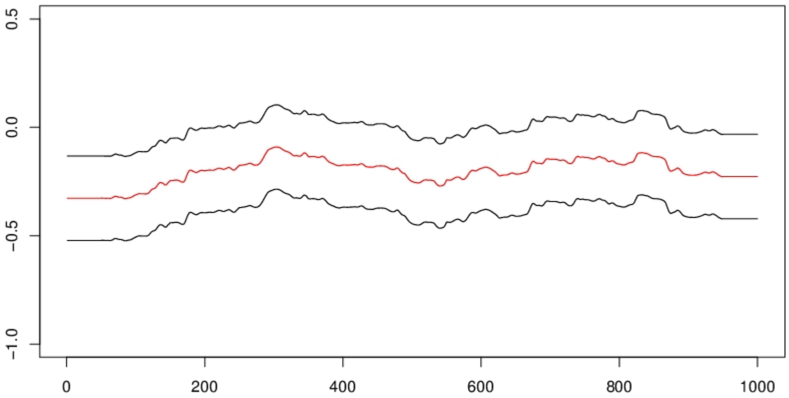

Figure 4.5: 0.95-confidence band of $\rho(u,1)$ of the wind data using the moving local TFT-bootstrap.

5 Proofs

5.1 Proofs of Section 2.5

We start by proving the tools of Section 2.5 as those are necessary for the proofs in Section 2.4.

Proof of Theorem 2.4. Let $MF^{\widetilde{X}}_{\lfloor uT\rfloor}(\lambda_l)$ resp. $MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)$ be defined as in Definition 2.2 but for the stationary time series $\widetilde{X}_t(u)=\sum_{j=-\infty}^{\infty}a(u,j)\,\varepsilon_{t-j}$ as in (2.6) resp. the i.i.d. sequence $\{\varepsilon_t\}$ (which is centered and has unit variance). From the proof of Theorem 10.3.1 in Brockwell and Davis (2006) it follows

$$MF^{\widetilde{X}}_{\lfloor uT\rfloor}(\lambda_l)=A(u,\lambda_l)\,MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)+R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l),$$

where $\mathrm{E}\,R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l)=0$ and by analogous arguments (using the 4th order Minkowski instead of the 2nd order Minkowski inequality)

$$\mathrm{E}\left|R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l)\right|^4\leqslant C\frac{1}{m^2}\left(\sum_{j=-\infty}^{\infty}|a(u,j)|\,\sqrt{\min(|j|,2m+1)}\right)^4,$$

where C only depends on $\mathrm{E}\,\varepsilon_1^4$ (but not u,m,T,l). Hence, it follows from $\sup_{0\leqslant u\leqslant 1}\sum_{j=-\infty}^{\infty}|a(u,j)|<\infty$ (by Definition 2.4 (c) (iii)) as in Brockwell and Davis, Proof of Theorem 10.3.1,

$$\sup_{u\in[0,1]}\sup_{l=1,\ldots,m}\mathrm{E}\,|R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l)|^4\to 0$$

as well as

$$\sup_{u\in[0,1]}\sup_{l=1,\ldots,m}\mathrm{E}\,|R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l)|^4=O\left(\frac{1}{m^2}\right)$$

under the stronger assumptions of (b).

In order to distinguish between the three cases in Definition 2.2, we introduce the notation

$$\zeta_t=\zeta_t(u,T)=\begin{cases}-1, & t-\mathrm{mod}(\lfloor uT\rfloor)\geqslant\lceil m/2\rceil,\\ 0 & -\lfloor m/2\rfloor\leqslant t-\mathrm{mod}(\lfloor uT\rfloor)<\lceil m/2\rceil,\\ +1, & t-\mathrm{mod}(\lfloor uT\rfloor)<-\lfloor m/2\rfloor.\end{cases}\tag{5.1}$$

Consider now

$$R^{(2)}_{\lfloor uT\rfloor,m}(\lambda_l) = MF_{\lfloor uT\rfloor}(\lambda_l) - MF^{\widetilde{X}}_{\lfloor uT\rfloor}(\lambda_l) = MF_{\lfloor uT\rfloor - \mathrm{mod}(\lfloor uT\rfloor)+l+\zeta_t m} - MF^{\widetilde{X}}_{\lfloor uT\rfloor}(\lambda_l)$$
$$= \frac{1}{\sqrt{2m+1}} \sum_{t=0}^{2m} \sum_{j=-\infty}^{\infty} \left(a_{t+\lfloor uT\rfloor - \mathrm{mod}(\lfloor uT\rfloor)+m\zeta_t+l-m,T}(j) - a(u,j)\right) e^{-it\lambda_l}$$
$$\cdot\, \varepsilon_{t+\lfloor uT\rfloor - \mathrm{mod}(\lfloor uT\rfloor)+m\zeta_t+l-m-j}$$

with $\mathrm{E}\, R^{(2)}_{\lfloor uT\rfloor,m}(\lambda_l) = 0$. First note that by Definition 2.2 (c) (i) and (ii) it holds uniformly in $0 \leqslant u \leqslant 1, 0 \leqslant t \leqslant 2m, 1 \leqslant l \leqslant m$

$$\left|a_{t+\lfloor uT\rfloor - \mathrm{mod}(\lfloor uT\rfloor)+m\zeta_t+l-m,T}(j) - a(u,j)\right| \leqslant K \frac{m}{T\, l(j)}.$$

Because $\{\varepsilon_t\}$ is i.i.d. and centered with fourth moments, it holds

$$\mathrm{E}\left(\varepsilon_{t_1-j_1}\varepsilon_{t_2-j_2}\varepsilon_{t_3-j_3}\varepsilon_{t_4-j_4}\right) = \begin{cases} O(1), & \text{the differences } t_s - j_s, s = 1,\dots,4, \text{ are pairwise equal} \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Consequently,

$$\sup_{u\in[0,1]} \sup_{l=1,\dots,m} \mathrm{E}\left|R^{(2)}_{\lfloor uT\rfloor,m}(\lambda_l)\right|^4 = O(1)\,\frac{m^4}{T^4}\left|\sum_{j=-\infty}^{\infty}\frac{1}{l(j)}\right|^4 = O\left(\frac{m^4}{T^4}\right) = O\left(\frac{1}{m^2}\right)$$

by Assumption A.2. Setting $R_{\lfloor uT\rfloor,m}(\lambda_l) = R^{(1)}_{\lfloor uT\rfloor,m}(\lambda_l) + R^{(2)}_{\lfloor uT\rfloor,m}(\lambda_l)$ completes the proof. ■

Proof of Corollary 2.1. We follow again the proof of Brockwell and Davis (2006), Theorem 10.3.1. By (2.7) decomposition (2.8) holds with

$$R'_{\lfloor uT\rfloor,m}(\lambda) = A(u,\lambda_l)\, MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)\overline{R_{\lfloor uT\rfloor,m}(\lambda_l)} + \overline{A(u,\lambda_l) MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)} R_{\lfloor uT\rfloor,m}(\lambda_l)$$
$$+ |R_{\lfloor uT\rfloor,m}(\lambda_l)|^2$$

with the notation of Theorem 2.4. Because

$$|A(u,\lambda)| \leqslant \sum_{j=-\infty}^{\infty} |a(u,j)| \leqslant K \sum_{j=-\infty}^{\infty} \frac{1}{l(j)} < \infty \tag{5.3}$$

uniformly in u and l by Definition 2.4 and $\mathrm{E}\left|MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)\right|^4 \leqslant C$ uniformly in u and l by (5.2), the assertions follow from Theorem 2.4 by an application of the Cauchy-Schwarz inequality. ■

5.2 Proofs of Section 2.4

Proof of Theorem 2.1. Assertion (a)(i) follows from the centeredness of $\{X_{t,T}\}$ in addition to the definition of the moving local Fourier coefficients.

For the other assertions, we get by Theorem 2.4 and $E(R_{\lfloor uT\rfloor,m}(\lambda_l)) = 0$

$$\mathrm{var}(MF_{\lfloor uT\rfloor}(\lambda_l))$$
$$= A(u,\lambda_l)\overline{A(u,\lambda_l)}\,\mathrm{var}(MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l))$$
$$+ A(u,\lambda_l)\,\mathrm{E}(MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)\,\overline{R_{\lfloor uT\rfloor,m}(\lambda_l)}) + \overline{A(u,\lambda_l)}\,\mathrm{E}(\overline{MF^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)}\,R_{\lfloor uT\rfloor,m}(\lambda_l)))$$
$$+ \mathrm{E}\left|R_{\lfloor uT\rfloor,m}(\lambda_l)\right|^2 =: A_1 + A_2 + \overline{A_2} + A_3.$$

Since $\{\varepsilon_t\}$ is i.i.d. with $\operatorname{var}\varepsilon_1 = 1$ a simple calculation shows $\operatorname{var}(M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)) = 1$, hence by (2.7) we get $A_1 = 2\pi f(u,\lambda_l)$. Consequently, by Theorem 2.4 and (5.3) applications of the Cauchy-Schwarz inequality yield $A_2 = o(1)$, $A_3 = o(1)$ and under the assumption of (b) even $A_2 = O(1/\sqrt{m})$ and $A_3 = O(1/m)$, which completes the proof. ■

Proof of Theorem 2.2. We will first prove the following corresponding result for the moving local Fourier coefficients $\{M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)\}$ of the i.i.d. innovation sequence $\{\varepsilon_t\}$:

$$\sup_{0\leqslant u\leqslant 1} \operatorname{cov}\left(M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l), M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)\right) = O\left(\frac{1}{|l-j|}\right) n, \tag{5.4}$$

$$\sup_{0\leqslant u\leqslant 1} \operatorname{cov}\left(M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l), M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)\right) = O\left(\frac{|l-j|}{m}\right) \quad \text{if } (l,j)\in\mathcal{M}(\lfloor uT\rfloor, m),\ l\neq j. \tag{5.5}$$

By Theorem 2.1 (a)(i) we get by the indepence of the innovations (and the existence of a second moment) with the notation as in (5.1)

$$\begin{aligned}
&\operatorname{cov}\left(M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l), M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j))\right) = E\left(M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l) M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)\right)\\
&= \frac{1}{2m+1}\sum_{t_1,t_2=0}^{2m} E(\varepsilon_{t_1-m+\lfloor uT\rfloor-\operatorname{mod}(\lfloor uT\rfloor)+m\zeta_l+l}\,\varepsilon_{t_2-m+\lfloor uT\rfloor-\operatorname{mod}(\lfloor uT\rfloor)+m\zeta_j+j})e^{-i\lambda_l t_1}e^{i\lambda_j t_2}\\
&= O\left(\frac{1}{2m+1}\sum_{t=\max\{0,j-l+m(\zeta_j-\zeta_l)\}}^{\min\{2m,2m+j-l+m(\zeta_j-\zeta_l)\}} e^{-i\lambda_{l-j}t-i\lambda_j(j-l+m(\zeta_j-\zeta_l))}\right)\\
&= O\left(\frac{1}{2m+1}\sum_{t=0}^{2m-|j-l+m(\zeta_j-\zeta_l)|} e^{-i\lambda_{l-j}t}\right) = O(1),
\end{aligned}$$

where the constants do not depend on u. By an application of Lemma A.4 in Kirch (2007) we get assertion (5.4). On the other hand for any $r > 0$ and $l \neq j$

$$\sum_{t=0}^{r} e^{i\lambda_{l-j}t} = -\sum_{t=r+1}^{2m} e^{i\lambda_{l-j}t} = O(2m-r),$$

which yields (5.5) if $\zeta_l = \zeta_j$, i.e. $(l,j)\in\mathcal{M}(\lfloor uT\rfloor, m)$.

The proof can now be concluded as in the proof of Theorem 2.1 by an application of Theorem 2.4 (for more details we refer to Lindner (2014), Proof of Theorem 5.3). ■

Proof of Theorem 2.3. We will first prove the following corresponding result for the moving local periodogram $\{M\!I^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)\}$ of the i.i.d. innovation sequence $\{\varepsilon_t\}$:

$$\operatorname{var}(M\!I^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l)) = 1 + \frac{\mathrm{E}(\varepsilon_1^4)-3}{2m+1} = O(1), \tag{5.6}$$

$$\operatorname{cov}(M\!I^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l), M\!I^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_j)) = R_u(l,j) + \tilde{R}_u(l,j),$$

$$\text{where } \sup_{0\leqslant u\leqslant 1}\sup_{1\leqslant l,j\leqslant m} |\tilde{R}_u(l,j)| = O\left(\frac{1}{m}\right), \tag{5.7}$$

$$\sup_{0\leqslant u\leqslant 1} |R_u(l,j)| = O\left(\frac{1}{|l-j|^2}\right), \tag{5.8}$$

$$\sup_{0\leqslant u\leqslant 1} |R_u(l,j)| = O\left(\frac{|l-j|^2}{m^2}\right) \quad \text{if } (l,j)\in\mathcal{M}(\lfloor uT\rfloor, m), l\neq j, \tag{5.9}$$

By $\mathrm{E}(\varepsilon_{k_1}\varepsilon_{k_2}) = 1_{\{k_1=k_2\}}$ it follows from the definition of the moving periodogram that

$$\mathrm{E}\, M\!I^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l) = \operatorname{var} M\!F^{\varepsilon}_{\lfloor uT\rfloor}(\lambda_l) = 1. \tag{5.10}$$

Furthermore, with the notation as in (5.1) we get by $\{\varepsilon_t\}$ i.i.d. and centered

$$\begin{aligned}
&E(MI^{\varepsilon}_{\lfloor uT\rfloor,m}(\lambda_l)\overline{MI^{\varepsilon}_{\lfloor uT\rfloor,m}(\lambda_j)})\\
&= \frac{1}{(2m+1)^2}\sum_{t_1,t_2,t_3,t_4=0}^{2m} e^{-i(t_1-t_2)\lambda_l}e^{i(t_3-t_4)\lambda_j}E(\varepsilon_{t_1+m\zeta_l+l}\varepsilon_{t_2+m\zeta_l+l}\varepsilon_{t_3+m\zeta_j+j}\varepsilon_{t_4+m\zeta_j+j})\\
&= \frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} e^{-i(t_1-t_2)\lambda_l}e^{i(t_3-t_4)\lambda_j}\,\mathrm{E}\left(\varepsilon_{t_1}\varepsilon_{t_2}\varepsilon_{t_3}\varepsilon_{t_4}\right)\\
&= \mathrm{E}\,\varepsilon_1^4\,\frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} 1_{\{t_1=t_2=t_3=t_4\}}\\
&\quad+\frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} 1_{\{t_1=t_2\neq t_3=t_4\}}\\
&\quad+\frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} 1_{\{t_1=t_4\neq t_2=t_3\}}e^{-it_1\lambda_{l+j}}e^{it_2\lambda_{l+j}}\\
&\quad+\frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} 1_{\{t_1=t_3\neq t_2=t_4\}}e^{-it_1\lambda_{l-j}}e^{it_2\lambda_{l-j}}\\
&= A_1+A_2+A_3+A_4.
\end{aligned}$$

Note that the number of overlapping summand for t_1, t_2 on the one hand and t_3, t_4 on the other hand is given by

$$\begin{aligned}
\Sigma(l,j) = \Sigma_{u,T}(l,j) &= 2m + \min(m\zeta_l + l, m\zeta_j + j) - \max(m\zeta_l + l, m\zeta_j + j) + 1\\
&= 2m + 1 - |l - j + m(\zeta_l - \zeta_j)|.
\end{aligned}$$

Consequently, we get

$$A_1 = \frac{\Sigma(l,j)}{(2m+1)^2}\,\mathrm{E}\,\varepsilon_1^4,$$

as well as

$$\begin{aligned}
A_2 &= 1 - \frac{1}{(2m+1)^2}\sum_{t_1,t_2=m\zeta_l+l}^{m\zeta_l+l+2m}\ \sum_{t_3,t_4=m\zeta_j+j}^{m\zeta_j+j+2m} 1_{\{t_1=t_2=t_3=t_4\}}e^{-i(t_1-t_2)\lambda_l}e^{i(t_3-t_4)\lambda_j}\\
&= 1 - \frac{\Sigma(l,j)}{(2m+1)^2}.
\end{aligned}$$

Similar arguments yield

$$\begin{aligned}
A_3 &= \frac{1}{(2m+1)^2}\left|\sum_{t=0}^{\Sigma(l,j)-1} e^{-it\lambda_{l+j}}\right|^2 - \frac{\Sigma(l,j)}{(2m+1)^2} = R_1 - \frac{\Sigma(l,j)}{(2m+1)^2},\\
A_4 &= \frac{1}{(2m+1)^2}\left|\sum_{t=0}^{\Sigma(l,j)-1} e^{-it\lambda_{l-j}}\right|^2 - \frac{\Sigma(l,j)}{(2m+1)^2} = R_2 - \frac{\Sigma(l,j)}{(2m+1)^2}.
\end{aligned}$$

As in the proof of Theorem 2.2 we conclude that R_2 fulfills (5.8) and (5.9) as well as that R_1 fulfills (5.9). Furthermore by Lemma A.4 in Kirch Kirch (2007) fulfills as $1 \leqslant l \neq j \leqslant m$

$$R_1 = O\left(\max\left(\frac{1}{(l+j)^2}, \frac{1}{(2m+1-(l+j))^2}\right)\right) = O\left(\frac{1}{|l-j|^2}\right).$$

Putting everything together we obtain the assertions with $R_u(l,j) = R_1 + R_2$ and $\tilde{R}_u(l,j) = -3\frac{\Sigma(l,j)}{(2m+1)^2}$.

For $l = j$ we have $\Sigma(l,j) = 2m+1$, hence $R_1 = 0$ and $R_2 = 1$, which implies (5.6).

The proof can now be concluded analogously to the proof of Theorem 2.1 by an application of Corollary 2.1. ■

5.3 Proofs of Section 2.6

Proof of Theorem 2.5. By (5.10) and Corollary 2.1 (a)(i) it holds

$$\mathrm{E}\left(\frac{1}{2m+1}\sum_{j=1}^{2m}\frac{MI_{\lfloor uT\rfloor,m}(\lambda_j)}{2\pi f(u,\lambda_j)}\right) \to 1.$$

Furthermore, by Theorem 2.3 (a) it holds

$$\operatorname{var}\left(\frac{1}{2m+1}\sum_{j=1}^{2m}\frac{MI_{\lfloor uT\rfloor,m}(\lambda_j)}{2\pi f(u,\lambda_j)}\right) = O\left(\frac{1}{m^2}\right)\sum_{i=1}^{2m}\sum_{j=1}^{2m}\left|\operatorname{cov}\left(MI_{\lfloor uT\rfloor,m}(\lambda_j), MI_{\lfloor uT\rfloor,m}(\lambda_j)\right)\right|$$
$$= O\left(\frac{a_m}{m}\right) + O\left(\frac{1}{m^2}\right)\sum_{\substack{i,j=1\\|i-j|\geqslant a_m}}^{2m}\left|\operatorname{cov}\left(MI_{\lfloor uT\rfloor,m}(\lambda_j), MI_{\lfloor uT\rfloor,m}(\lambda_j)\right)\right| = o(1)$$

with any choice of $a_m \to \infty$ such that $a_m/m \to 0$. An application of the Cauchy-Schwarz inequality now yields the assertion. ■

Proof of Theorem 2.6. To ease notation, we set $MI_s(\lambda) = MI_{s,m}(\lambda)$. The key observation for the proof is the following: Due to the compactness of the Kernel the only non-zero summands in $\hat{f}(k)$ (for every k) are the ones with $|t - \mathrm{mod}(k)| \leqslant Cmh$ for a suitable constant C. Consequently, the only moving local periodograms involved in every estimation (with h sufficiently small) are from the middle case in Definition 2.2 for the moving local periodogram at k. Furthermore, for every $k \in \{\lfloor uT\rfloor - \lceil \frac{m}{2}\rceil + 1, \ldots, \lfloor uT\rfloor + \lfloor \frac{m}{2}\rfloor\}$ it either holds $MI_k(\lambda_t) = MI_s(\lambda_t)$ or $MI_k(\lambda_t) = MI_{s+m}(\lambda_t)$, where $s = s(u,m) = mj(u,m) + \lfloor m/2\rfloor + 1$ for a suitably chosen $0 \leqslant j(u,m) < T/m$ with $|s - \lfloor uT\rfloor| \leqslant m$. This can be seen by the way the moving periodograms are constructed: Moving to the right, we calculate the frequencies $\lambda_1, \ldots, \lambda_m$, then jump to λ_1 again. In each estimation $\hat{f}(k)$ only moving periodograms from neighboring observations and with close frequencies are used, i.e. only moving periodograms within the same sequence $\lambda_1, \ldots, \lambda_m$ are used. For any u and all k from the corresponding range only two (consecutive) such sequences are involved, which are characterized by $s(u,m)$ (the point in the middle of each sequence) above.

By Definition 2.4 and (2.7) it holds $\sup_{j=1,\ldots,m}|f(u,\lambda_j) - f(s/T,\lambda_j)| \to 0$, so that by Theorem 2.1 (b)

$$\sup_{1\leqslant l\leqslant m}|\mathrm{E}\, MI_s(\lambda_l) - 2\pi f(u,\lambda_l)| \to 0. \tag{5.11}$$

By Assumptions A.3 and the uniform Lipschitz condition on the time-varying spectral density it follows

$$\sup_{1\leqslant l\leqslant m}\left|\mathrm{E}\,\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)M\!I_s(\lambda_t)-2\pi f(u,\lambda_l)\right|$$
$$\leqslant \sup_{1\leqslant l\leqslant m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)\left(\mathrm{E}\,M\!I_s(\lambda_t)-2\pi f(u,\lambda_t)\right)\right|$$
$$+2\pi\sup_{1\leqslant l\leqslant m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)\left(f(u,\lambda_t)-f(u,\lambda_l)\right)\right|=o(1)+O(h)=o(1).$$

as well as an analogous assertion for s replaced by $s+m$.

Consequently, it is sufficient to prove that for any $s=s(u,m)$ it holds

$$\sup_{1\leqslant l\leqslant m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)\left(M\!I_s(\lambda_t)-\mathrm{E}\,M\!I_s(\lambda_t)\right)\right|\to 0.$$

From Theorem 2.3 we get by the choice $a_m=\sqrt{m}$ for $t_1\neq t_2$

$$\operatorname{cov}(M\!I_s(\lambda_{t_1}),M\!I_s(\lambda_{t_2}))|=O\left(\frac{1}{\sqrt{m}}\right) \tag{5.12}$$

and boundedness for $t_1=t_2$.

The remainder of the proof is very close to the proof of Theorem A1 in Franke and Härdle (1992) and as such is only sketched. Define $\alpha_m:=\frac{h}{m^{\frac{1}{4}}}$, as well as $\mu_m:=\left\lfloor\frac{1}{\alpha_m}\right\rfloor$.

Define $\theta_j=\pi j/\mu_m$, $j=0,\ldots,\mu_m$. Then

$$\sup_{1\leqslant l\leqslant m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)\left(M\!I_s(\lambda_t)-\mathrm{E}\,M\!I_s(\lambda_t)\right)\right|$$
$$=\sup_{j=0,\ldots,\mu_m-1}\ \sup_{|\lambda_l-\theta_j|\leqslant\pi\alpha_m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\lambda_l-\lambda_t\right)\left(M\!I_s(\lambda_t)-\mathrm{E}\,M\!I_s(\lambda_t)\right)\right|$$
$$\leqslant\sup_{j=0,\ldots,\mu_m-1}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\theta_j-\lambda_t\right)\left(M\!I_s(\lambda_t)-\mathrm{E}\,M\!I_s(\lambda_t)\right)\right|$$
$$+\sup_{|\lambda_l-\lambda_r|\leqslant\pi\alpha_m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}\left(K_h\left(\lambda_l-\lambda_t\right)-K_h\left(\lambda_r-\lambda_t\right)\right)M\!I_s(\lambda_t)\right|$$
$$+\sup_{|\lambda_l-\lambda_r|\leqslant\pi\alpha_m}\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}\left(K_h\left(\lambda_l-\lambda_t\right)-K_h\left(\lambda_r-\lambda_t\right)\right)\mathrm{E}(M\!I_s(\lambda_t))\right|$$
$$=A_1+A_2+A_3.$$

By (5.12) and the Markov-inequality it holds

$$P(A_1\geqslant\varepsilon)\leqslant\sum_{j=0}^{\mu_m-1}P\left(\left|\frac{1}{2m+1}\sum_{t=-m}^{2m}K_h\left(\theta_j-\lambda_t\right)\left(M\!I_s(\lambda_t)-\mathrm{E}\,M\!I_s(\lambda_t)\right)\right|\geqslant\varepsilon\right)$$
$$\leqslant\sum_{j=0}^{\mu_m-1}\frac{1}{(2m+1)^2\varepsilon^2}\sum_{t_1,t_2=-m}^{2m}K_h\left(\theta_j-\lambda_{t_1}\right)K_h\left(\theta_j-\lambda_{t_2}\right)\operatorname{cov}\left(M\!I_s(\lambda_{t_1}),M\!I_s(\lambda_{t_2})\right)$$
$$=O\left(\frac{\mu_m}{hm}+\frac{\mu_m}{\sqrt{m}}\right)=O\left(\frac{1}{h^2\,m^{3/4}}+\frac{1}{hm^{1/4}}\right)=o(1).$$

By Theorem 2.5 and Assumptions A.3

$$A_2 = O\left(\frac{\alpha_m}{h^2}\right) \frac{1}{2m+1} \sum_{t=-m}^{2m} MI_s(\lambda_t) = O\left(\frac{\alpha_m}{h^2}\right) = O\left(\frac{1}{hm^{1/4}}\right) = o(1),$$

where we used the fact that f is uniformly bounded, hence

$$\frac{1}{2m+1} \sum_{t=-m}^{2m} MI_s(\lambda_t) = O\left(\frac{1}{2m+1} \sum_{t=-m}^{2m} \frac{MI_s(\lambda_t)}{2\pi f(u,\lambda_t)}\right) = O(1).$$

Similarly, by (5.11) and Assumptions A.3 it holds

$$A_3 = O\left(\frac{\alpha_m}{h^2}\right) = o(1),$$

completing the proof. ∎

5.4 Proofs of Section 3.1

Proof of Theorem 3.1. First by definition it holds $\mathrm{E}^* X_t^* = 0$ as well as uniformly in $k \in \{\lfloor uT \rfloor - \lceil \frac{m}{2} \rceil + 1, \ldots, \lfloor uT \rfloor + \lfloor \frac{m}{2} \rfloor\}$

$$\mathrm{cov}^*(MF_k^*, MF_l^*) = \begin{cases} 2\pi f(u, \lambda_{\mathrm{mod}(k)}) + o_P(1), & k = l, \\ 0, & \text{else.} \end{cases}$$

Consequently,

$$\begin{aligned} &\mathrm{cov}^*(MF^*_{\lfloor uT \rfloor}(\lambda_r), MF^*_{\lfloor uT \rfloor + h}(\lambda_s)) \\ &= \begin{cases} 2\pi f(u,\lambda_r) + o_P(1), & r = s \text{ and } MF^*_{\lfloor uT \rfloor}(\lambda_r) = MF^*_{\lfloor uT \rfloor + h}(\lambda_r), \\ 0, & \text{else.} \end{cases} \end{aligned}$$

The condition $MF^*_{\lfloor uT \rfloor}(\lambda_r) = MF^*_{\lfloor uT \rfloor + h}(\lambda_s)$ is only met if the same bootstrap moving Fourier coefficients are used, i.e. if they are the closest moving Fourier coefficients to frequency λ_k for both locations $\lfloor uT \rfloor$ as well as $\lfloor uT \rfloor + h$. Consequently, the condition is met for the moving local Fourier coefficients belonging to the moving Fourier coefficients MF_k^* with $\lfloor uT \rfloor + h - \lfloor m/2 \rfloor \leqslant k < \lfloor uT \rfloor + \lceil m/2 \rceil$ for $h \geqslant 0$ resp. $\lfloor uT \rfloor - \lfloor m/2 \rfloor \leqslant k < \lfloor uT \rfloor + h + \lceil m/2 \rceil$ for $h < 0$.

We will now add the missing variances of the Fourier coefficients that are not overlapping in order to be able to argue as in the stationary case. Additionally we prove that the contribution by these non-overlapping coefficients is asymptotically negligible:

Since $\mathrm{cov}^*(MF_k^*, \overline{MF_l^*}) = 0$, $k \neq l$, we get uniformly

$$\begin{aligned} &(2m+1)\,\mathrm{Cov}^*(X^*_{\lfloor uT \rfloor, T}, X^*_{\lfloor uT \rfloor + h, T}) = (2m+1)\,\mathrm{E}^*\left(X^*_{\lfloor uT \rfloor, T} X^*_{\lfloor uT \rfloor + h, T}\right) \\ &= o_P(m) + 2\pi \sum_{l=1}^{m} f(u,\lambda_l) e^{i\lambda_l \lfloor uT \rfloor} e^{-i\lambda_l(\lfloor uT \rfloor + h)} + 2\pi \sum_{l=1}^{m} f(u,\lambda_l) e^{-i\lambda_l \lfloor uT \rfloor} e^{i\lambda_l(\lfloor uT \rfloor + h)} \\ &\quad + (2m+1)\, R_{m,h} \\ &= o_P(m) + 2\pi \sum_{l=-m}^{m} f(u,\lambda_l) e^{-i\lambda_l h} - 2\pi f(u,0) + (2m+1)\, R_{m,h} \\ &= o_P(m) + (2m+1) \int_{-\pi}^{\pi} f(u,\lambda) e^{-i\lambda h}\, d\lambda + (2m+1)\, R_{m,h} \\ &= o_P(m) + (2m+1)\, c(u,h) + (2m+1)\, R_{m,h}, \end{aligned}$$

where $R_{m,h}$ contains those variances belonging to the moving local Fourier coefficients that are different at time $\lfloor uT \rfloor$ and $\lfloor uT \rfloor + h$. Let us denote the set of moving local Fourier coefficients that are different by $A_{u,h}$, where for the following arguments it is only important that the cardinality of $A_{u,h}$ is $|h|$ and that the indices in it are (at most) two subsequent sets of numbers, so that by Lemma A.4 in Kirch (2007)

$$\sum_{l \in A_{u,h}} e^{-i\lambda_l(h+n)} = O\left(|h|\right) \text{ as well as}$$

$$\sum_{l \in A_{u,h}} e^{-i\lambda_l(h+n)} = O\left(\max\left(\frac{2m+1}{|n+h|}, \frac{2m+1}{2m+1-|n+h|}\right)\right) \quad \text{for } 1 \leqslant |h+n| \leqslant 2m$$

and analogous assertions for the term involving $e^{+i\lambda_l(h+n)}$.

With the first assertion and $f(u,\lambda_l) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c(u,n)\, e^{-i\lambda_l n}$ we get

$$\sup_{|h| \leqslant m/\log(m)} |R_{m,h}| = \sup_{|h| \leqslant m/\log(m)} \left| \frac{1}{2m+1} \sum_{n=-\infty}^{\infty} c(u,n) \left(\sum_{l \in A_{u,n}} e^{-i\lambda_l(n+h)} + \sum_{l \in A_{u,n}} e^{i\lambda_l(n+h)} \right) \right|$$
$$= O\left(\frac{1}{\log m}\right) \sum_{n=-\infty}^{\infty} |c(u,n)| = o(1)$$

by the summability of the local autocovariance function. Furthermore,

$$\sup_{m/\log(m) < |h| \leqslant m} |R_{m,h}|$$
$$\leqslant \sum_{|n| \geqslant \sqrt{m}} |c(u,n)| + O(1) \sup_{m/\log(m) < |h| \leqslant m} \sum_{|n| < \sqrt{m}} |c(u,n)| \max\left(\frac{1}{|n+h|}, \frac{1}{2m+1-|n+h|}\right) = o(1)$$

as the first summand converges to zero by the summability of the local autocovariance function and the second summand by the fact that $|n+h| \geqslant m/\log m - \sqrt{m} \to \infty$ as well as $2m+1-|n+h| \geqslant 2m+1-m-\sqrt{m} \to \infty$. This completes the proof. ∎

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